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ON THE MEAN ROTATION TENSORS

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Abstract—In this paper, we criticize a serious essential defect of the famous Cauchy's measure of mean rotation of a deformable body formulated by Zheng and Hwang (1988, *Chinese Sci. Bull.* **33**, 1705–1707; (English translation, 1989) **34**, 897–901; 1992, *ASME J. Appl. Mech.* **59**, 505–510). A number of mean rotation tensors are proposed, which are prime generalizations of Cauchy's mean rotation and avoid the defect of the latter. In this frame, several new significant geometrical meanings of the finite rotation tensor **Q** in the polar decomposition of the deformation gradient **F** are revealed. The so-called large rotation tensor $\mathbf{R}_w$, as a quite natural generalization of the infinitesimal rotation tensor $\mathbf{W} = (\mathbf{F}-\mathbf{F}^T)/2$, is introduced and is a very good approximation of **Q** in the case of small or moderate strain accompanied by large rotation. A short discussion on the rates of mean rotation, the role of **Q** in constitutive equations, and the effect of choosing a reference configuration is provided. Finally, we investigate the global measures of mean rotation and the global kinetic equations of a finite deforming body.

1. INTRODUCTION

The theory of finite rotation of a deformable body has recently attracted many authors' attention, because it has been well recognized (see, for example, Dienes, 1979, 1987; Dafalias, 1983, 1987, 1988, and many consequent papers; Zheng, 1990, 1992) that rotations and their rates (i.e. spins) play an important role in describing complex irreversible mechanical behaviour of (especially, anisotropic) materials. However, the problem of measuring finite rotation of a deformable body is more difficult than that of a rigid body, just as commented by Truesdell and Toupin (1960, p. 273) "The theory of finite rotation has always presented singular difficulty, although the essential idea is simple." A dominant problem is that rotation of a deformable body is in a sense a mean rotation since the line elements radiating from the same material point cannot in general be described in terms of a unified rotation tensor.

Suppose that a three-dimensional deformable body undergoes a finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where **X** and **x** are the position vectors of the typical material particle of the deformable body with respect to the reference configuration and the current (at time t) configuration of the deformable body, respectively. Denote by $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ the deformation gradient and $\mathbf{F} = \mathbf{QU}$, the right polar decomposition of **F** where **Q** is a rotation tensor called the finite rotation tensor and **U** is the right stretch tensor. For a given direction (i.e. a unit vector) **r**, the angle through which the projection upon the **r**-plane (i.e. the plane which is normal to the direction **r**) of a line element d**X** rotated right-handed about the axis **r** to the projection upon the **r**-plane of $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$, is denoted by $\theta(\mathbf{r}-d\mathbf{X})$ as delineated in Fig. 1. Since $\theta(\mathbf{r}, d\mathbf{X})$ depends only on the direction $\mathbf{N} = d\mathbf{X}/|d\mathbf{X}|$ of the linear element $d\mathbf{X}: \theta(\mathbf{r}, d\mathbf{X}) \equiv \theta(\mathbf{r}, \mathbf{N})$, Zheng and Hwang (1988, 1992) called $\theta(\mathbf{r}, \mathbf{N})$ the deformation rotation angle of the direction **N** about the axis **r**.

The set of all directions perpendicular to **r** is denoted by U_r. We found the following three measures of mean rotation in the literature:

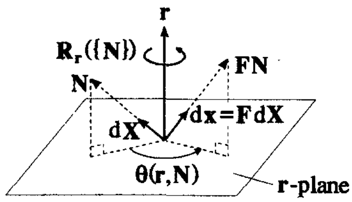


Fig. 1. The deformation rotation angle $\theta(\mathbf{r}, \mathbf{N})$ and the mean rotation tensor of the direction set $\{\mathbf{N}\}$.

$$\text{Cauchy (1841)} \quad \chi_r^o = \text{the mean value of } \theta(\mathbf{r}, \mathbf{N}) \text{ for all } \mathbf{N} \in U_r, \tag{1a}$$

$$\text{Novozhilov (1948)} \quad \tan \nu_r = \text{the mean value of } \tan \theta(\mathbf{r}, \mathbf{N}) \text{ for all } \mathbf{N} \in U_r, \tag{1b}$$

$$\text{Marzano (1987)} \quad \cos \mu_r = \text{the mean value of } \cos \theta(\mathbf{r}, \mathbf{N}) \text{ for all } \mathbf{N} \in U_r. \tag{1c}$$

In this paper, we call χ_r^o the original Cauchy's mean rotation angle with respect to $\mathbf{r}$. In spite of the elegance of Cauchy's concept of mean rotation, the analytical expression of χ_r^o in terms of $\mathbf{F}$ and $\mathbf{r}$ had never been found until the recent works by Zheng (1987) and Zheng and Hwang (1988, 1992; see also Martins and Podio-Guidugli, 1992). Before these recent works, Novozhilov's expression for $\tan \nu_r$ [cf. (9)] was widely accepted as the measure of mean rotation in finite deformation.

U denotes the full direction set which consists of all spatial directions and S denotes a subset of U. The observation that U_r is a proper subset of U gives rise to the following definition (Zheng, 1987, 1989):

$$\chi_r^o(S) = \text{the mean value of } \theta(\mathbf{r}, \mathbf{N}) \text{ for all } \mathbf{N} \in S, \tag{2}$$

called the extended Cauchy's mean rotation angle of S. In particular, one may suggest that the extended Cauchy's mean rotation angle $\chi_r^o(U)$ of the full direction set U would be a more meaningful measure of mean rotation than the original one χ_r^o. Unfortunately, we found (Zheng, 1989) that Cauchy's concept and its foregoing generalization (2) has a serious essential defect, as briefly explained in the next section.

Since the effect of the finite rotation tensor $\mathbf{Q}$ and a strain measure in the constitutive functional for a simple material (Noll, 1958) can simply be separated, $\mathbf{Q}$ is widely understood as the best one among the measures of rotation of a deformable body having undergone finite deformation. Dienes (1979, 1987) assumed that the relative spin $\boldsymbol{\Omega}_p = \dot{\mathbf{Q}}\mathbf{Q}^T$, rather than the material spin $\boldsymbol{\Omega}$ (i.e. the antisymmetric part of the velocity gradient), should occur the privileged position in the formulation of rate-type constitutive equations. This work has given rise to lengthy discussions, arguments and counter-arguments in recent years. In the works of Zheng (1990, 1992), a constitutive equivalence principle was established which can be used to formulate constitutive laws precisely from their arotational forms for simple materials based upon the objective axiom only. As a consequence, Dienes' assumption and the importance of both the relative spin $\boldsymbol{\Omega}_p$ and the finite rotation tensor $\mathbf{Q}$ itself are solidly supported.

Thus, tensorial algorithms of the finite rotation tensor $\mathbf{Q}$ is meaningful and were studied by Hoger and Carlson (1984). The complete and precise tensorial algorithms of $\mathbf{Q}$ were obtained firstly by Xiong and Zheng (1988). Since the complexity of the precise expressions of $\mathbf{Q}$, an approximate but simple expression of $\mathbf{Q}$ is necessary and useful in order to construct, especially, the non-linear theories of rods, plates, and shells in moderate or large rotation deformation.

In this paper, a number of mean rotation tensors are proposed which are prime natural generalizations of Cauchy's measure of mean rotation and do not possess the defect of the latter. In this frame we may marry the finite rotation tensor $\mathbf{Q}$ with some new significant geometric meanings in addition to those illustrated by Grioli (1940), Martins and Podio-Guidugli (1979, 1980), and Zheng and Hwang (1988, 1992). The so-called large rotation tensor $\mathbf{R}_w$ and the maximum Cauchy's mean rotation tensor $\mathbf{R}_{max}$ are of particular interest.

We note that $\mathbf{R}_w$ is a quite natural generalization of $\mathbf{W} = (\mathbf{F}-\mathbf{F}^T)/2$ which describes the rotation when the deformation is infinitesimal. As the deformation is of small or moderate strain accompanied by large rotation, two new approximate but quite simple algorithms for $\mathbf{Q}$ are constituted in terms of $\mathbf{R}_w$, which may be taken as a new basis for describing the kinematics of the non-linear theories of rods, plates, and shells. A short discussion about the rates of mean rotation, the role of the finite rotation tensor $\mathbf{Q}$ in constitutive equations, and the effect of choosing a reference configuration is provided. Finally, we investigate the global measures of mean rotation and the global kinetic equations of a finite deforming body in order to describe the global behaviour rather than the local behaviour of a deforming body.

Tensor algebra is the most convenient tool to use in the present analysis, and for background readers may refer to the books of Bowen and Wang (1976), Gurtin (1981), etc. For example, the scalar, vector and tensor products of any two vectors $\mathbf{a}$ and $\mathbf{b}$ are denoted by $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}$, respectively, the prefix tr and the superscript T indicate trace and transpose, respectively, and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ and $|\mathbf{A}| = \sqrt{\operatorname{tr}\mathbf{A}^T\mathbf{A}}$ represent the norms for any vector $\mathbf{a}$ and second-order tensor $\mathbf{A}$. All components of vectors and tensors are referred to a rectangular Cartesian coordinate system $\{X_1, X_2, X_3\}$.

2. A SERIOUS DEFECT OF CAUCHY'S MEASURE OF MEAN ROTATION

To formulate the original Cauchy's mean rotation angle χ_r°, Zheng and Hwang (1988, 1992; see also Zheng, 1987) established the following very simple analytical expressions for χ_r:

$$T_r \sin\chi_r = \mathbf{w}\cdot\mathbf{r}, \tag{3a}$$

$$T_r \cos\chi_r = \tfrac{1}{2}(\operatorname{tr}\mathbf{F}-\mathbf{r}\cdot\mathbf{F}\mathbf{r}) = 1+\tfrac{1}{2}(\operatorname{tr}\mathbf{E}-\mathbf{r}\cdot\mathbf{E}\mathbf{r}), \tag{3b}$$

with $T_r \geqslant 0$ and they showed that $\chi_r^\circ = \chi_r$. In (3), the additive decomposition of $\mathbf{F}$:

$$\mathbf{F} = \mathbf{I}+\mathbf{E}+\mathbf{W}, \tag{4a}$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{F}-\mathbf{F}^T), \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{F}+\mathbf{F}^T)-\mathbf{I} \tag{4b}$$

is used, where $\mathbf{I}$ denotes the second-order identity tensor and $\mathbf{w}$ the axial vectors of $\mathbf{W}$: that is, $\mathbf{W} = \mathbf{w}\times\mathbf{I}$, or $\mathbf{W}\mathbf{u} = \mathbf{w}\times\mathbf{u}$ for any vector $\mathbf{u}$, or

$$W_{32} = -W_{23} = w_1, \quad W_{13} = -W_{31} = w_2, \quad W_{21} = -W_{12} = w_3. \tag{5}$$

We explain that Cauchy's measure of mean rotation possesses a serious defect. For this sake, we choose a rectangular Cartesian coordinate system $\{X_i\}$ so that X_3-coordinate coincides with the $\mathbf{r}$ direction and denote by $\mathbf{N} = (\cos\varphi, \sin\varphi, 0)$ with $0 \leqslant \varphi < 2\pi$ a unit vector on the $\mathbf{r}$-plane. Thus, the deformation rotation angle $\theta(\mathbf{r},\mathbf{N})$ can be expressed in the following form (Zheng and Hwang, 1992; see also Zheng, 1993):

$$\rho\cos\theta(\mathbf{r},\mathbf{N}) = x_r + R_r\cos 2(\varphi-\varphi_r), \tag{6a}$$

$$\rho\sin\theta(\mathbf{r},\mathbf{N}) = y_r - R_r\sin 2(\varphi-\varphi_r), \tag{6b}$$

in which $\rho \geqslant 0$,

$$x_r = T_r\cos\chi_r = \tfrac{1}{2}(F_{11}+F_{22}), \tag{7a}$$

$$y_r = T_r\sin\chi_r = \tfrac{1}{2}(F_{21}-F_{12}), \tag{7b}$$

$$2T_r = \sqrt{(F_{11}+F_{22})^2+(F_{21}-F_{12})^2}, \tag{7c}$$

and

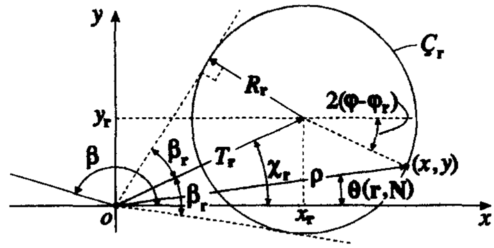


Fig. 2. Deformation rotation circle $Ç_{\mathbf{r}}$ and Cauchy's mean rotation angle $\chi_{\mathbf{r}}$.

$$R_{\mathbf{r}} \cos 2\varphi_{\mathbf{r}} = \tfrac{1}{2}(F_{11}-F_{22}), \tag{8a}$$

$$R_{\mathbf{r}} \sin 2\varphi_{\mathbf{r}} = \tfrac{1}{2}(F_{21}+F_{12}), \tag{8b}$$

$$2R_{\mathbf{r}} = \sqrt{(F_{11}-F_{22})^2+(F_{21}+F_{12})^2}. \tag{8c}$$

With the notation given above in (7) and (8), the exact formula of Novozhilov's measure of mean rotation is of the form (Zheng and Hwang, 1992):

$$\tan \nu_{\mathbf{r}} = \operatorname{sign}(x_{\mathbf{r}}) \frac{y_{\mathbf{r}}}{\sqrt{x_{\mathbf{r}}^2 - R_{\mathbf{r}}^2}}, \tag{9}$$

while Novozhilov (1948) missed the factor of the sign of $\chi_{\mathbf{r}}$ in (9).

By introducing a plane rectangular Cartesian coordinate system $\{x, y\}$, the curve

$$Ç_{\mathbf{r}} = \{(x(\varphi), y(\varphi)) : 0 \leqslant \varphi < 2\pi\}, \tag{10}$$

for $x(\varphi) = \rho \cos \theta(\mathbf{r}, \mathbf{N})$ and $y(\varphi) = \rho \sin \theta(\mathbf{r}, \mathbf{N})$ according to (6) is obviously two superimposed circles on the xOy plane with the center $(x_{\mathbf{r}}, y_{\mathbf{r}})$ and the radius $R_{\mathbf{r}}$ and was named the deformation rotation circle by Zheng and Hwang (1988, 1992), as delineated in Fig. 2. Furthermore, the original point O in Fig. 2 is outside, on, or inside the deformation rotation circle $Ç_{\mathbf{r}}$ if and only if $Đ_{\mathbf{r}}$ defined by

$$Đ_{\mathbf{r}} = T_{\mathbf{r}}^2 - R_{\mathbf{r}}^2 = \tfrac{1}{2}\operatorname{tr}\mathbb{D} - \mathbf{r}\cdot\mathbb{D}\mathbf{r} = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} = F_{11}F_{22} - F_{21}F_{12}, \tag{11a}$$

$$\mathbb{D} = \tfrac{1}{2}[(\operatorname{tr}\mathbf{F})(\mathbf{F}+\mathbf{F}^{\mathrm{T}}) - \mathbf{F}^2 - (\mathbf{F}^2)^{\mathrm{T}}], \tag{11b}$$

is positive, null, or negative.

We emphasize that from Fig. 2, the $\chi_{\mathbf{r}}$ formulated in (3) can be explained as:

$$\chi_{\mathbf{r}} = \text{the geometric average of } \theta(\mathbf{r}, \mathbf{N}) \text{ for all } \mathbf{N} \in U_{\mathbf{r}}, \tag{12}$$

in comparing with $\chi_{\mathbf{r}}^{\circ}$ as the algebraic average (i.e. mean value) of $\theta(\mathbf{r}, \mathbf{N})$ for all $\mathbf{N} \in U_{\mathbf{r}}$. In the sequel, we shall validate that $\chi_{\mathbf{r}}^{\circ}$ is not always equal to $\chi_{\mathbf{r}}$. Therefore, we call $\chi_{\mathbf{r}}$ in this paper the (generalized) Cauchy's mean rotation angle with respect to $\mathbf{r}$.

For instance, consider a homogeneous deformation for which the matrix of the deformation gradient $\mathbf{F}$ is of the form:

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}. \tag{13a}$$

One can easily check that the determinant of $\mathbf{F}$ is positive and

$$Ð_1 = F_{22}F_{33} - F_{32}F_{23} = 1, \tag{13b}$$

$$Ð_2 = F_{33}F_{11} - F_{13}F_{31} = 0, \tag{13c}$$

$$Ð_3 = F_{11}F_{22} - F_{21}F_{12} = -1. \tag{13d}$$

Substituting (13a) into (7)–(9) yields the Novozhilov's and Cauchy's mean rotation angles v_i and χ_i about the X_i-axes for $i = 1, 2, 3$ as follows:

$$v_1 = \pi/2, \quad v_2 = \arctan\left(\tfrac{0}{0}\right) = \text{indefinite}, \quad v_3 = \arctan\left(\frac{1}{\sqrt{-5}}\right) = \text{no sense}, \tag{13e}$$

$$\chi_1 = \arctan\left(\tfrac{2}{5}\right) \approx 0.3805\,\pi, \quad \chi_2 = 0, \quad \chi_3 = \arctan\left(\tfrac{1}{2}\right) \approx 0.4636\,\pi. \tag{13f}$$

In general, from (9) and Fig. 2, we see that if the y-axis touches or crosses the deformation rotation circle $Ç_r$, then v_r is of no sense. According to the algorithm provided by Xiong and Zheng (1988), we have calculated the finite rotation tensor **Q** and the right stretch tensor **U**. From them, the principal values Λ_1, Λ_2, Λ_3 of **U**, the maximum deformation rotation angle γ_{max} (Zheng and Hwang, 1992) caused by the strain only and the rotation angle Θ of **Q** can be given as follows:†

$$\Lambda_1 \approx 6.265, \quad \Lambda_2 \approx 1.269, \quad \Lambda_3 \approx 0.3774, \tag{13g}$$

$$\gamma_{max} = \arcsin\left(\frac{\Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_3}\right) \approx 0.3468\,\pi, \tag{13h}$$

$$\Theta \approx 0.5205\,\pi. \tag{13i}$$

Let us keep the example shown above in (13) in mind and go on our general analysis. Consider the following three possibilities.

(i) The original point O in Fig. 2 is outside the deformation rotation circle $Ç_r$, i.e. $T_r > R_r$ or $Ð_r > 0$, as exampled in (13b). In this case, for any given angle β in the interval

$$Æ = (\chi_r + \beta_r, \quad 2\pi + \chi_r - \beta_r), \tag{14}$$

where $\beta_r = \arcsin(R_r/T_r)$ (see Fig. 2), $\theta(\mathbf{r}, \mathbf{N})$ as a mapping from U_r to the angle region $(\beta - 2\pi, \beta]$ is continuous; and the relation $\chi_r^o = \chi_r$ holds and is invariant for any choice of $\beta \in Æ$. In Fig. 3(a), we plot the $Ç_r$ and the relation between $\theta(\mathbf{r}, \mathbf{N})$ and $\mathbf{N} = (0, \cos\varphi, \sin\varphi)$ for the example (13) with respect to $\mathbf{r} = (1, 0, 0)$.

(ii) The original point O in Fig. 2 is on the deformation rotation circle $Ç_r$, i.e. $T_r = R_r$ or $Ð_r = 0$, as exampled in (13c). In this case $\theta(\mathbf{r}, \mathbf{N})$ is discontinuous at the original point O. However, for any given angle β in the interval

$$Æ' = (\chi_r + \pi/2, \chi_r + 3\,\pi/2), \tag{15}$$

if the angle region of $\theta(\mathbf{r}, \mathbf{N})$ is stipulated as $(\beta - 2\pi, \beta]$, then the relation $\chi_r^o = \chi_r$ still holds and is invariant for any choice of $\beta \in Æ'$. In Fig. 3(b), we plot the $Ç_r$ and the relation between $\theta(\mathbf{r}, \mathbf{N})$ and $\mathbf{N} = (\sin\varphi, 0, \cos\varphi)$ for the example (13) with respect to $\mathbf{r} = (0, 1, 0)$.

(iii) As shown in (13d), it is possible that the original point O in Fig. 2 is inside the deformation rotation circle $Ç_r$, i.e. $T_r < R_r$ or $Ð_r < 0$. Thus, it is not able to stipulate an angle interval as the angle region of $\theta(\mathbf{r}, \mathbf{N})$ so that $\theta(\mathbf{r}, \mathbf{N})$ would be a continuous function on $Ç_r$, and the calculation for χ_r^o has no invariance in choosing an angle region for $\theta(\mathbf{r}, \mathbf{N})$.

† The facts $Ð_3 < 0$ from (13d) and $\Theta + \gamma_{max} = 0.8673\pi < \pi$ from (13h) and (13i) destroy the second theorem in the paper of Zheng and Hwang (1992) since this "theorem" implies a false consequence that if $\Theta + \gamma_{max} < \pi$, then with respect to any given direction **r** the negative x-axis in Fig. 2 would not touch or cross the deformation rotation circle $Ç_r$.

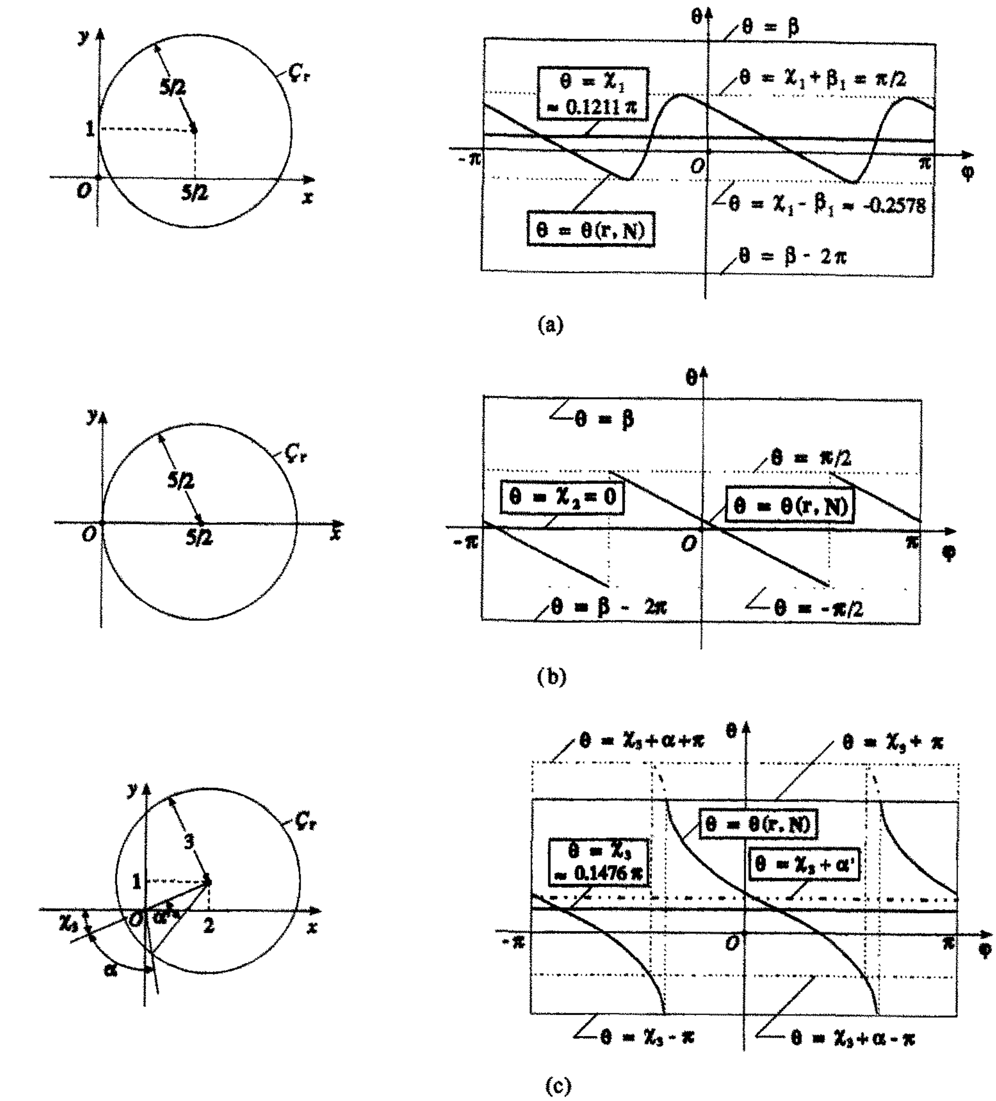


Fig. 3. The deformation rotation angle $\theta(\mathbf{r}, \mathbf{N})$ as a function of $\mathbf{N}$ on $\mathcal{Ç}_r$ and its mean value: (a) $Đ_r > 0$, (b) $Đ_r = 0$, (c) $Đ_r < 0$.

In fact, a explicit complex analysis performed by Zheng (1989) showed that if the angle region of $\theta(\mathbf{r}, \mathbf{N})$ is stipulated as:

$$\theta(\mathbf{r}, \mathbf{N}) \in (-\pi + \chi_r + \alpha, \pi + \chi_r + \alpha], \tag{16a}$$

for an arbitrarily given angle α, then the calculation of the original Cauchy's mean rotation angle χ_r° can be expressed as follows [see Fig. 3(c)]:

$$\chi_r^\circ = \chi_r + \alpha', \tag{16b}$$

$$\frac{R_r}{\sin(\pi - \alpha)} = \frac{T_r}{\sin(\alpha - \alpha')}, \quad \text{or}\, \alpha' = \alpha - \arcsin\left(\frac{T_r}{R_r}\sin\alpha\right). \tag{16c}$$

The relation $\chi_r^\circ = \chi_r$ holds, if and only if χ_r is known previously and the angle region for $\theta(\mathbf{r}, \mathbf{N})$ is taken as either $(-\pi + \chi_r, \pi + \chi_r]$ or $[-\pi + \chi_r, \pi + \chi_r)$. In Fig. 3(c), we plot the $\mathcal{Ç}_r$ and the relation between $\theta(\mathbf{r}, \mathbf{N})$ and $\mathbf{N} = (\cos\varphi, \sin\varphi, 0)$ for the example (13) with respect to $\mathbf{r} = (0, 0, 1)$.

The analysis given above in cases (i), (ii) and (iii) is an abstract of the detailed complex analysis performed by Zheng (1989). Truesdell and Toupin (1960, p. 276) pointed out that Cauchy failed to notice that, because of the equivocality of $\theta(\mathbf{r}, \mathbf{N})$, his definition (1a) is not sufficient to calculate $\chi_{\mathbf{r}}^{\circ}$. In more detail, we have seen that the original Cauchy's measure $\chi_{\mathbf{r}}^{\circ}$ of mean rotation fails in case (iii), and the measure $\chi_{\mathbf{r}}$ is really a generalization of the original one $\chi_{\mathbf{r}}^{\circ}$. Therefore, we should marry the generalized Cauchy's mean rotation angle $\chi_{\mathbf{r}}$ with a new explicit geometric meaning rather than both the vague one (12) and the original one (1a). This gives rise to new definitions of mean rotation tensors studied in the next section.

For an infinitesimal deformation, i.e. $|\mathbf{F}-\mathbf{I}| \ll 1$, from (11) and (7a) one can easily write

$$Ð_{\mathbf{r}} \approx 1+(\operatorname{tr}\mathbf{E}-\mathbf{r}\cdot\mathbf{E}\mathbf{r}) > 0, \tag{17a}$$

$$x_{\mathbf{r}} = 1+\tfrac{1}{2}(\operatorname{tr}\mathbf{E}-\mathbf{r}\cdot\mathbf{E}\mathbf{r}) > 0. \tag{17b}$$

Thus, the deformation rotation circle $Ç_{\mathbf{r}}$ for any direction $\mathbf{r}$ is located on the right-hand half plane $\{(x, y) : x > 0\}$; and for the most acceptable angle region $(-\pi, \pi)$ of $\theta(\mathbf{r}, \mathbf{N})$, the relation $\chi_{\mathbf{r}}^{\circ} = \chi_{\mathbf{r}}$ always holds.

However, even though for an infinitesimal deformation, to calculate the extended Cauchy's mean rotation angle $\chi_{\mathbf{r}}^{\circ}$ (U) of the full direction set U defined in (2), problems similar to that encountered in case (iii) is, in general, not able to be avoided and $\chi_{\mathbf{r}}^{\circ}$ (U) is thus indefinite. Fortunately, in the frame of mean rotation tensor studied in the next section, the mean rotation tensors of the set U of all spatial directions are well defined and formulated.

3. MEAN ROTATION TENSORS

As is well-known (see, e.g. Guo, 1981; Xiong and Zheng, 1989a), the rotation tensor $\mathbf{R}$ about a direction $\mathbf{r}$ right-handed through an angle θ can be expressed in the following canonical form:

$$\mathbf{R} = \cos\theta\mathbf{I}+\sin\theta\mathbf{r}\times\mathbf{I}+(1-\cos\theta)\mathbf{r}\otimes\mathbf{r}. \tag{18}$$

In particular, we denote by $\mathbf{R}_{\mathbf{r}}$ the rotation tensor about $\mathbf{r}$ through the angle $\chi_{\mathbf{r}}$, namely

$$\mathbf{R}_{\mathbf{r}} = \cos\chi_{\mathbf{r}}\mathbf{I}+\sin\chi_{\mathbf{r}}\mathbf{r}\times\mathbf{I}+(1-\cos\chi_{\mathbf{r}})\mathbf{r}\otimes\mathbf{r}, \tag{19}$$

which was called by Zheng and Hwang (1992) the Cauchy's mean rotation tensor about the direction $\mathbf{r}$.

Suppose that the direction set S of concern is measurable in a sense. The function

$$\Delta(\mathbf{R}, S) = \int_{\mathbf{N}\in S} |\mathbf{FN}-\mathbf{RN}|^2 = \operatorname{tr}[(\mathbf{F}-\mathbf{R})\mathbf{K}(S)(\mathbf{F}-\mathbf{R})^T] \tag{20}$$

for any rotation tensor $\mathbf{R}$ describes the total square deviation between the actual deformation $\{\mathbf{FN} : \mathbf{N}\in S\}$ and a rigid rotation $\{\mathbf{RN} : \mathbf{N}\in S\}$ of the direction set S, where

$$\mathbf{K}(S) = \int_{\mathbf{N}\in S} \mathbf{N}\otimes\mathbf{N}. \tag{21}$$

An explanation about $\mathbf{K}(S)$ is required. In addition to the direction sets $U_{\mathbf{r}}$ and U considered previously, the following direction sets:

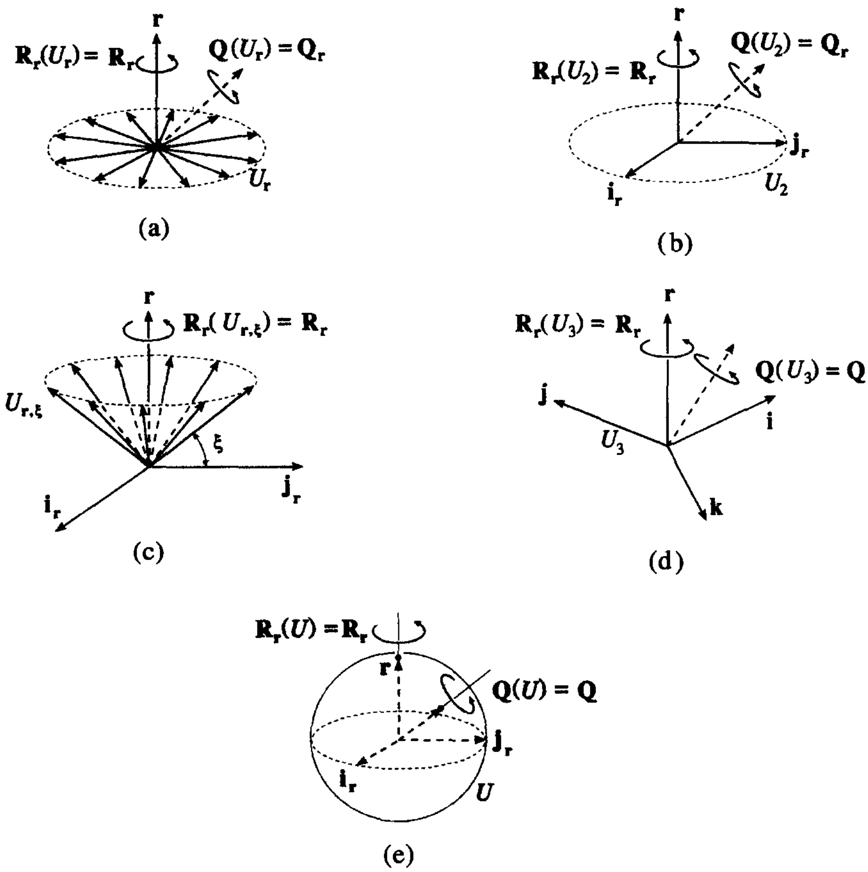


Fig. 4. Direction sets and their mean rotation tensors: (a) U_r, (b) U_2, (c) $U_{r,\xi}$, (d) U_3, (e) U.

$U_2 = \{\mathbf{i}_r, \mathbf{j}_r\}$ consists of two orthogonal directions $\mathbf{i}_r$ and $\mathbf{j}_r$ perpendicular to $\mathbf{r}$, (22a)

$U_{r,\xi} = \{\mathbf{N} \in U : \mathbf{N} \cdot \mathbf{r} = \sin\xi\}, -\frac{1}{2}\pi < \xi < \frac{1}{2}\pi$, is a one-dimensional manifold, (22b)

$U_3 = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal triad (i.e. three orthogonal directions) (22c)

are also of interest, as depicted in Fig. 4. If a typical spatial direction $\mathbf{N}$ is expressed in the form:

$$\mathbf{N} = \mathbf{N}(\psi, \xi) = (\cos\psi\, \mathbf{i}_r + \sin\psi\, \mathbf{j}_r)\cos\xi + \sin\xi\, \mathbf{r}, \quad (0 \leqslant \psi < 2\pi, \ -\pi/2 \leqslant \xi \leqslant \pi/2), \tag{23}$$

we may define the following $\mathbf{K}(S)$ with respect to the most natural measures in finite sets, one- and two-dimensional manifolds, respectively:

$$\mathbf{K}(U_2) = \mathbf{i}_r \otimes \mathbf{i}_r + \mathbf{j}_r \otimes \mathbf{j}_r = \mathbf{I} - \mathbf{r} \otimes \mathbf{r}, \tag{24a}$$

$$\mathbf{K}(U_r) = \int_0^{2\pi} \mathbf{N}(\psi, 0) \otimes \mathbf{N}(\psi, 0)\, \mathrm{d}\psi = \pi(\mathbf{I} - \mathbf{r} \otimes \mathbf{r}), \tag{24b}$$

$$\mathbf{K}(U_{r,\xi}) = \int_0^{2\pi} \mathbf{N} \otimes \mathbf{N} \cos\xi\, \mathrm{d}\psi = \pi\cos^3\xi\, [\mathbf{I} + (2\tan^2\xi - 1)\mathbf{r} \otimes \mathbf{r}], \tag{24c}$$

$$\mathbf{K}(U_3) = \mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k} = \mathbf{I}, \tag{24d}$$

$$\mathbf{K}(U) = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\pi} \mathbf{N} \otimes \mathbf{N}\, \mathrm{d}\psi \right) \cos\xi\, \mathrm{d}\xi = \frac{4\pi}{3}\mathbf{I}. \tag{24e}$$

A glance at (24) shows that these symmetric tensors $\mathbf{K}(S)$ for $S = U_2$, U_r, $U_{r,\xi}$, U_3 and U can be expressed in the following unified form:

$$\mathbf{K}(S) = k(\mathbf{I}+\tau\mathbf{r}\otimes\mathbf{r}), \quad (k > 0, \tau \geqslant -1). \tag{25}$$

One can easily list more direction sets which possess above property (25).

Denote by $\mathfrak{R}_r$ the set of all rotation tensors about the given axis $\mathbf{r}$. The rotation tensor $\mathbf{R}_r(S)$, at which $\Delta(\mathbf{R}, S)$ as a function of $\mathbf{R}$ in the domain $\mathfrak{R}_r$ is the least possible, is of particular interest. From (3), (18), (20) and (25), it can easily be derived that

$$\frac{\partial\Delta(\mathbf{R}, S)}{\partial\theta} = 4k\, T_r \sin(\theta-\chi_r), \tag{26a}$$

$$\frac{\partial^2\Delta(\mathbf{R}, S)}{\partial\theta^2} = 4k\, T_r \cos(\theta-\chi_r) \tag{26b}$$

for $S = U_2$, U_r, $U_{r,\xi}$, U_3 and U. Thus, from the minimum conditions $\partial\Delta(\mathbf{R}, S)/\partial\theta = 0$ and $\partial^2\Delta(\mathbf{R}, S)/\partial\theta^2 \geqslant 0$ we arrive at $\theta = \chi_r + 2m\pi$ for $m = 0, \pm 1, \pm 2, \ldots$, and finally

$$\mathbf{R}_r(S) = \mathbf{R}_r, \quad (\text{for } S = U_2, U_r, U_{r,\xi}, U_3, U), \tag{27}$$

where $\mathbf{R}_r$ is the Cauchy's mean rotation tensor about $\mathbf{r}$ formulated by (3) and (19).

The foregoing analysis enlightens us on the definition that for any direction set S the rotation tensor $\mathbf{R}_r(S)$ is named the generalized Cauchy's mean rotation tensor of S about the axis $\mathbf{r}$. Thus, as shown in (27) and illustrated in Fig. 4, we have already proved the following theorems.

Theorem 1. The Cauchy's mean rotation tensor $\mathbf{R}_r$ about the axis $\mathbf{r}$ can be explained as the generalized Cauchy's mean rotation tensors $\mathbf{R}_r(S)$ for $S = U_2$, U_r, $U_{r,\xi}$, U_3 and U.

As a new version of the third theorem given by Zheng and Hwang (1992), the following theorem can easily be verified:

Theorem 2. If $T_r \neq 0$ and $\bar{D}_r \neq 0$, then the polar decomposition

$$(\mathbf{I}-\mathbf{r}\otimes\mathbf{r})\mathbf{F}(\mathbf{I}-\mathbf{r}\otimes\mathbf{r}) = \hat{\mathbf{R}}\hat{\mathbf{U}} \tag{28}$$

for a rotation tensor $\hat{\mathbf{R}}$ and a symmetric tensor $\hat{\mathbf{U}}$ with tr $\hat{\mathbf{U}} \geqslant 0$ exists and is unique; and $\hat{\mathbf{R}}$ is identified with the Cauchy's mean rotation tensor $\mathbf{R}_r$.

One can also prove that the generalized Cauchy's mean rotation tensor $\mathbf{R}_r(\{\mathbf{N}\})$ of the set of a single spatial direction $\mathbf{N}$ is just

$$\mathbf{R}_r(\{\mathbf{N}\}) = \cos\theta(\mathbf{r},\mathbf{N})\mathbf{I}+\sin\theta(\mathbf{r},\mathbf{N})\mathbf{r}\times\mathbf{I}+[1-\cos\theta(r,\mathbf{N})]\mathbf{r}\otimes\mathbf{r}, \tag{29}$$

as shown in Fig. 1, with $\theta(\mathbf{r},\mathbf{N})$ the deformation rotation angle of $\mathbf{N}$ about the axis $\mathbf{r}$.

Denote by $\mathfrak{R}$ the set of all three-dimensional rotation tensors. Similar to the definition of the generalized Cauchy's mean rotation tensor $\mathbf{R}_r(S)$, the rotation tensor $\mathbf{Q}(S)$, at which $\Delta(\mathbf{R}, S)$ as a function of $\mathbf{R}$ in the domain $\mathfrak{R}$ is the least possible, is defined as the (local) mean rotation tensor of the direction set S. For $S = U_2$, U_r, $U_{r,\xi}$, U_3 and U, introduce

$$\mathbf{F}^* = \mathbf{F}(\mathbf{I}+\tau\mathbf{r}\otimes\mathbf{r}) = \mathbf{Q}^*\mathbf{U}^*, \quad (\mathbf{U}^* = \sqrt{\mathbf{F}^{*T}\mathbf{F}^*}) \tag{30}$$

where $\tau \geqslant -1$ is a parameter corresponding to (25), $\mathbf{Q}^*\mathbf{U}^*$ is the right polar decomposition of $\mathbf{F}^*$ with $\mathbf{Q}^*$ a rotation tensor and $\mathbf{U}^*$ a positive (if $\tau > -1$) or semi-positive (if $\tau = -1$) definite symmetric tensor which is uniquely determined by the relation $\mathbf{U}^{*2} = \mathbf{F}^{*T}\mathbf{F}^*$. The uniqueness of $\mathbf{Q}^*$ as a solution from (30) is obvious as $\tau > -1$, and will be proved in the sequel as $\tau = -1$.

Substituting (30) and (25) into (20) yields

$$\Delta(\mathbf{R},S) = k\{3+\tau+\mathrm{tr}\,\mathbf{F}^T\mathbf{F}+\tau\,|\,\mathbf{Fr}\,|^2-2\mathrm{tr}\,(\mathbf{R}^T\mathbf{Q}^*\mathbf{U}^*)\}. \tag{31}$$

Let $\mathbf{U}^* = \upsilon_1\mathbf{m}_1\otimes\mathbf{m}_1+\upsilon_2\mathbf{m}_2\otimes\mathbf{m}_2+\upsilon_3\mathbf{m}_3\otimes\mathbf{m}_3$, with υ_1, υ_2, $\upsilon_3\geqslant 0$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ an orthonormal triad, denote the spectral form of $\mathbf{U}^*$. From the inequality

$$\mathrm{tr}\,(\mathbf{R}^T\mathbf{Q}^*\mathbf{U}^*) = \sum_{i=1}^{3}\upsilon_i\mathbf{m}_i\cdot(\mathbf{R}^T\mathbf{Q})\mathbf{m}_i \leqslant \sum_{i=1}^{3}\upsilon_i = \mathrm{tr}\,\mathbf{U}^* \tag{32}$$

it follows that in general if and only if $\mathbf{R}^T\mathbf{Q}^* = \mathbf{I}$, $\Delta(\mathbf{R}, S)$ arrives at its minimum point for all $\mathbf{R}\in\mathfrak{R}$. Thus, we have

Theorem 3. For $S = U_2$, U_r, $U_{r,\xi}$, U_3 and U, the mean rotation tensor $\mathbf{Q}(S)$ of S is just the rotation tensor $\mathbf{Q}^*$ in the polar decomposition $\mathbf{F}^* = \mathbf{F}\,(\mathbf{I}+\tau\mathbf{r}\otimes\mathbf{r}) = \mathbf{Q}^*\mathbf{U}^*$.

Because of $\mathbf{F}^* = \mathbf{F}$ for both $S = U_3$ and $S = U$, as a consequence of above theorem we can further state the following theorem which elucidates two new significant geometric meanings, as delineated in Figs 4(d) and (e), of the finite rotation tensor $\mathbf{Q}$ in addition to those illustrated by Grioli (1940), Martins and Podio-Guidugli (1979, 1980), and Zheng and Hwang (1992) [cf. Theorem 6(ii)].

Theorem 4. The finite rotation tensor $\mathbf{Q}$ in the polar decomposition of the deformation gradient can be referred to as being the mean rotation tensors of both the full direction set U and any orthonormal triad $S_3 = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; that is, $\mathbf{Q}(U) = \mathbf{Q}(S_3) = \mathbf{Q}$.

Now we turn our attention to proving that when $\tau = -1$ (for example, $S = S_2$ or $S = S_r$), as a rotation tensor-valued solution from the polar decomposition:

$$\mathbf{F}_r = \mathbf{F}(\mathbf{I}-\mathbf{r}\otimes\mathbf{r}) = \mathbf{Q}_r\mathbf{U}_r, \quad (\mathbf{U}_r = \sqrt{\mathbf{C}_r}, \mathbf{C}_r = \mathbf{F}_r^T\mathbf{F}_r) \tag{33}$$

$\mathbf{Q}_r$ is unique. Since $\mathbf{I}_r = \mathbf{I}-\mathbf{r}\otimes\mathbf{r}$ is the identity tensor restricted in the $\mathbf{r}$-plane and $\mathbf{C}_r$ is a positive definite symmetric tensor restricted in the $\mathbf{r}$-plane, we can formulate $\mathbf{U}_r$ in a similar form to that in the two-dimensional problem formulated, for example, by Hoger and Carlson (1984). A tensorial algorithm of $\mathbf{U}_r$ and $\mathbf{R}_r$ from $\mathbf{F}_r$ can be given below:

$$I_r = \mathrm{tr}\,\mathbf{C}_r, \quad II_r = \tfrac{1}{2}[I_r^2-tr\,(\mathbf{C}_r^2)], \tag{34a}$$

$$\mathbf{U}_r = (\mathbf{C}_r+\sqrt{II_r}\mathbf{I}_r)/\sqrt{I_r+2\sqrt{II_r}}, \tag{34b}$$

$$\mathbf{Q}_r\mathbf{I}_r = (\sqrt{I_r+2\sqrt{II_r}}\mathbf{I}_r-\mathbf{U}_r)/\sqrt{II_r}, \tag{34c}$$

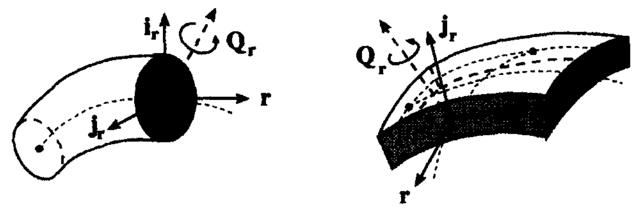
$$(\mathbf{Q}_r\mathbf{r})\times\mathbf{I} = (\mathbf{Q}_r\mathbf{I}_r)(\mathbf{r}\times\mathbf{I})(\mathbf{Q}_r\mathbf{I}_r)^T, \tag{34d}$$

$$\mathbf{Q}_r = \mathbf{Q}_r\mathbf{I}_r+(\mathbf{Q}_r\mathbf{r})\otimes\mathbf{r}, \tag{34e}$$

which confirms consequently the uniqueness of $\mathbf{Q}_r$. This completes the following Theorem 5, as portrayed in Figs 4(a) and (b).

Theorem 5. The rotation tensor $\mathbf{Q}_r$ in the polar decomposition of $\mathbf{F}(\mathbf{I}-\mathbf{r}\otimes\mathbf{r})$ is unique and can be explained as the mean rotation tensors $\mathbf{Q}(S)$ of both the set $S = U_r$ of all directions on the $\mathbf{r}$-plane and the set $S = U_2$ of any two orthogonal directions perpendicular to $\mathbf{r}$.

In applications, we may refer to $\mathbf{Q}_r$, instead of $\mathbf{Q}$, as being the measure of finite rotation of a cross-section with normal direction $\mathbf{r}$ of a rod, plate, or shell which undergoes finite



(a) (b)

Fig. 5. The mean rotation tensors of cross-sections: (a) rod, (b) shell.

deformation as $\mathbf{r}$ is tangent to the axial curviline of the rod [Fig. 5(a)] or the middle surface of the plate or shell [Fig. 5(b)].

Incidentally, it can easily be proved that all the three mean rotation tensors $\mathbf{Q}_{N(1)}$, $\mathbf{Q}_{N(2)}$, and $\mathbf{Q}_{N(3)}$ with respect to the three principal directions $\mathbf{N}(1)$, $\mathbf{N}(2)$, and $\mathbf{N}(3)$ of the right stretch tensor $\mathbf{U}$ (or the right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$) are just equal to the finite rotation tensor $\mathbf{Q}$.

Finally, consider the finite rotation tensor $\mathbf{Q}$ in another local sense. Denote by $N(\mathbf{X}, \varepsilon) = \{\mathbf{X}+\delta\mathbf{X} : |\delta\mathbf{X}| < \varepsilon\}, 0 < \varepsilon \ll 1$, a ball neighborhood of the typical point $\mathbf{X}$. We measure the deviation between the actual local deformation (described by $\mathbf{F}$) and a local rigid rotation (described by a rotation tensor $\mathbf{R}$) of $N(\mathbf{X}, \varepsilon)$ by

$$\int_{N(\mathbf{X},\varepsilon)} |\mathbf{F}\delta\mathbf{X}-\mathbf{R}\delta\mathbf{X}|^2 = \frac{4\pi}{15}\varepsilon^5 \operatorname{tr}(\mathbf{F}-\mathbf{R})^T(\mathbf{F}-\mathbf{R}). \tag{35}$$

Thus, similar to Theorems 4 and 5 we have:

Theorem 6. (i) Among all local rigid rotations about the fixed axis $\mathbf{r}$, the one produced by the Cauchy's mean rotation tensor $\mathbf{R}_r$ is most closed to the actual local deformation; and (ii) among all local rigid rotations, the one caused by the finite rotation tensor $\mathbf{Q}$ is most closed to the actual local deformation.

Similar conclusions to Theorem 6 (ii) were given by Grioli (1940) and Martins and Podio-Guidugli (1979, 1980). Theorems 1–6 link the rotation tensors $\mathbf{Q}$, $\mathbf{R}_r$, $\mathbf{Q}_r$ and $\mathbf{Q}^*$ in the polar decompositions of $\mathbf{F}$, $(\mathbf{I}-\mathbf{r}\otimes\mathbf{r})\mathbf{F}(\mathbf{I}-\mathbf{r}\otimes\mathbf{r})$, $\mathbf{F}_r = \mathbf{F}(\mathbf{I}-\mathbf{r}\otimes\mathbf{r})$ and $\mathbf{F}^* = \mathbf{F}(\mathbf{I}+\tau\mathbf{r}\otimes\mathbf{r})$ with plentiful and significant geometric meanings.

4. SOME SPECIAL MEAN ROTATION TENSORS

For a given deformation gradient $\mathbf{F}$, Cauchy's mean rotation tensor $\mathbf{R}_r$ as a tensor function of a unit vector $\mathbf{r}$ is a mapping from U to $\mathfrak{R}$. Denote by $\mathbf{p}$ the axis direction of the finite rotation tensor $\mathbf{Q}$ so that the rotation angle Θ of $\mathbf{Q}$ fulfils (see, e.g. Guo, 1981) $0 \leqslant \Theta \leqslant \pi$, i.e.

$$\mathbf{Q} = \cos\Theta\mathbf{I} + \sin\Theta\mathbf{p}\times\mathbf{I} + (1-\cos\Theta)\mathbf{p}\otimes\mathbf{p}, \quad (0 \leqslant \Theta \leqslant \pi). \tag{36}$$

Denote by $\mathring{\mathbf{w}} = \mathbf{w}/|\mathbf{w}|$ the direction of $\mathbf{w}$ which is the axial vector of the antisymmetric part $\mathbf{W}$ (i.e. $\mathbf{W} = \mathbf{w}\times\mathbf{I}$) of $\mathbf{F}$. The direction $\mathbf{r}_{max}$ is introduced in the sense:

$$\chi_{max} = \chi_{\mathbf{r}_{max}} = \max\{\chi_{\mathbf{r}} : \text{for all } \mathbf{r}\in U\}, (0 \leqslant \chi_{max} \leqslant \pi). \tag{37}$$

Of course, among Cauchy's mean rotation tensors with respect to all spatial directions, the ones with respect to $\mathbf{p}$, $\mathring{\mathbf{w}}$ and $\mathbf{r}_{max}$ are of particular interest.

In the work of Zheng and Hwang (1988, 1992), $\mathbf{Q}$ was recognized as the Cauchy's mean rotation tensor about $\mathbf{p}$, so that we have

$$T_{\mathbf{p}} \cos\Theta = 1+\frac{1}{2}(\operatorname{tr}\mathbf{E}-\mathbf{p}\cdot\mathbf{E}\mathbf{p}), \tag{38a}$$

$$T_{\mathbf{p}} \sin\Theta = \mathbf{p}\cdot\mathbf{w}, \tag{38b}$$

with $T_{\mathbf{p}} \geqslant 0$. Since $\Delta(\mathbf{R}, U) = (4\pi/3)\operatorname{tr}(\mathbf{F}-\mathbf{R})^T(\mathbf{F}-\mathbf{R})$ as a function of $\mathbf{R}$ in the domain $\mathfrak{R}$ is the least possible (see Theorem 4) at $\mathbf{R} = \mathbf{Q}$, from (18) and (20) we conclude that Θ and $\mathbf{p}$ have to satisfy the following additional equation

$$\not{p}_{\mathbf{p}}\mathbf{p} = \sin\Theta\mathbf{w}+(1-\cos\Theta)\mathbf{E}\mathbf{p}, \quad (\not{p}_{\mathbf{p}} \geqslant 0). \tag{39}$$

Equations (38) and (39) constitute a new algorithm for Θ, $\mathbf{p}$ and $\mathbf{Q}$.

Denote by $\mathbf{R}_{\max}$ the Cauchy's mean rotation tensor with respect to $\mathbf{r}_{\max}$, namely,

$$\mathbf{R}_{\max} = \cos\chi_{\max}\mathbf{I}+\sin\chi_{\max}\mathbf{r}_{\max}\times\mathbf{I}+(1-\cos\chi_{\max})\mathbf{r}_{\max}\otimes\mathbf{r}_{\max}, \tag{40}$$

called the maximum Cauchy's mean rotation tensor. From (3) it yields that the direction $\mathbf{r}_{\max}$ obeys

$$\not{p}_{\max}\mathbf{r}_{\max} = \cos\chi_{\max}\mathbf{w}+\sin\chi_{\max}\mathbf{E}\mathbf{r}_{\max}, \quad (\not{p}_{\max} \geqslant 0). \tag{41}$$

The difference between the equations (39) and (41) fulfilled by $\mathbf{p}$ and $\mathbf{r}_{\max}$, respectively, shows that the finite rotation tensor $\mathbf{Q}$ is, in general, not the maximum Cauchy's mean rotation tensor $\mathbf{R}_{\max}$.

Since in the theory of infinitesimal deformation the rotation is described in terms of $\mathbf{W}$ or $\mathbf{w}$, to deal with a finite deformation problem we pay naturally special attention to the Cauchy's mean rotation tensor, say $\mathbf{R}_w$, about the direction $\mathring{\mathbf{w}} = \mathbf{w}/|\mathbf{w}|$ of $\mathbf{w}$, i.e.

$$\mathbf{R}_{\mathbf{w}} = \cos\chi_{\mathbf{w}}\mathbf{I}+\sin\chi_{\mathbf{w}}\mathring{\mathbf{w}}\times\mathbf{I}+(1-\cos\chi_{\mathbf{w}})\mathring{\mathbf{w}}\otimes\mathring{\mathbf{w}}, \tag{42}$$

where $\chi_{\mathbf{w}}$ is the generalized Cauchy's mean rotation angle with respect to $\mathring{\mathbf{w}}$ and is formulated according to (3) as follows:

$$T_{\mathbf{w}} \sin\chi_{\mathbf{w}} = |\mathbf{w}|, \tag{43a}$$

$$T_{\mathbf{w}} \cos\chi_{\mathbf{w}} = 1+\frac{1}{2}(\operatorname{tr}\mathbf{E}-\mathring{\mathbf{w}}\cdot\mathbf{E}\mathring{\mathbf{w}}), \tag{43b}$$

with $T_{\mathbf{w}} \geqslant 0$. Obviously, $\mathbf{R}_{\mathbf{w}}$ is in general neither $\mathbf{Q}$ nor $\mathbf{R}_{\max}$. We call $\mathbf{R}_{\mathbf{w}}$ the large rotation tensor with good reasons explained in the next section.

We employ an example to further detail the differences among $\mathbf{Q}$, $\mathbf{R}_{\max}$ and $\mathbf{R}_{\mathbf{w}}$. Consider a finite homogeneous deformation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ whose rectangular Cartesian component form is supposed to be of the form:

$$x_1 = (3\cos\Theta-2\sin\Theta)X_1+(2\cos\Theta-4\sin\Theta)X_2-(\sin\Theta)X_3, \tag{44a}$$

$$x_2 = (3\sin\Theta+2\cos\Theta)X_1+(2\sin\Theta+4\cos\Theta)X_2+(\cos\Theta)X_3, \tag{44b}$$

$$x_3 = X_2+2X_3. \tag{44c}$$

It is evident that the finite rotation tensor $\mathbf{Q}$ is a rotation about the X_3-axis through the angle $\Theta = \Theta(t)$. The principal stretches Λ_1, Λ_2 and Λ_3 can be calculated as follows:

$$\Lambda_1 \cong 5.686, \quad \Lambda_2 \cong 2.424, \quad \Lambda_3 \cong 0.8900, \tag{44d}$$

which indicate that the strain is quite large. In order to calculate the maximum Cauchy's mean rotation angle $\chi_{\max}$, we propose to employ the following recurrence procedure

$$\tan\chi_n = \frac{2\mathbf{w}\cdot\mathbf{r}}{2+\operatorname{tr}\mathbf{E}-\mathbf{r}_n\cdot\mathbf{E}\mathbf{r}_n}, \tag{44e}$$

$$\mathbf{r}_{n+1} = (1-\mu)\mathbf{r}_n+\mu\frac{\mathbf{w}+\mathbf{E}\mathbf{r}_n\tan\chi_n}{|\mathbf{w}+\mathbf{E}\mathbf{r}_n\tan\chi_n|}, \tag{44f}$$

$$\chi_{\max} = \lim_{n\to\infty}\chi_n,\quad \mathbf{r}_{\max} = \lim_{n\to\infty}\mathbf{r}_n, \tag{44g}$$

for $n = 0, 1, 2, \ldots$, by taking $\mathbf{r}_0 = \mathring{\mathbf{w}}$. In (44f), μ $(0 < \mu \leqslant 1)$ is a proper parameter introduced in order to improve the convergency in (44g). Some values of $\chi_{\max}$, χ_w, α_w (the angle between $\mathbf{p}$ and $\mathring{\mathbf{w}}$) and $\alpha_{\max}$ (the angle between $\mathbf{p}$ and $\mathbf{w}$) as functions of Θ are listed in Table 1.

Denote by ε, $0 \leqslant \varepsilon \ll 1$, an infinitesimal parameter and f_ε and g_ε two scalar-, vector-, or second-order tensor-valued functions of ε. If there exist constants $K > 0$ and c such that $|f_\varepsilon - g_\varepsilon| < K\varepsilon^c$ as $\varepsilon \to 0$, then we employ the notation

$$f_\varepsilon - g_\varepsilon = \mathrm{O}(\varepsilon^c) \quad\text{or}\quad f_\varepsilon = g_\varepsilon + \mathrm{O}(\varepsilon^c). \tag{45}$$

Suppose that the deformable body undergoes an infinitesimal deformation, i.e. $\mathbf{F} = \mathbf{I}+\mathrm{O}(\varepsilon)$, or both $\mathbf{E} = \mathrm{O}(\varepsilon)$ and $\mathbf{W} = \mathrm{O}(\varepsilon)$. In this case we prove that differences among the three measures $\mathbf{Q}$, $\mathbf{R}_w$ and $\mathbf{R}_{\max}$ become negligible. In fact, one can obtain

$$\chi_r = \mathbf{w}\cdot\mathbf{r}+\mathrm{O}(\varepsilon^2), \tag{46a}$$

$$\chi_w = |\mathbf{w}|+\mathrm{O}(\varepsilon^2), \tag{46b}$$

$$\Theta = |\mathbf{w}|+\mathrm{O}(\varepsilon^2),\quad \mathbf{p} = \mathring{\mathbf{w}}+\mathrm{O}(\varepsilon), \tag{46c}$$

$$\chi_{\max} = |\mathbf{w}|+\mathrm{O}(\varepsilon^2),\quad \mathbf{r}_{\max} = \mathring{\mathbf{w}}+\mathrm{O}(\varepsilon). \tag{46d}$$

Thus, $\mathbf{w}$ is just the mean rotation vector of the infinitesimal deformation; and if omitting an error of order $\mathrm{O}(\varepsilon^2)$, then we have $\mathbf{Q} = \mathbf{R}_w = \mathbf{R}_{\max}$.

Let $\{X, Y, Z\}$ be a rectangular Cartesian coordinate system and u, v, and w the displacement components of an infinitesimal deformation. A simple analysis based on (34) can result in that the mean rotation of the Z-plane corresponds to the following components:

$$\frac{\partial w}{\partial Y},\quad -\frac{\partial w}{\partial Z},\quad \frac{1}{2}\left(\frac{\partial v}{\partial X}-\frac{\partial u}{\partial Y}\right) \tag{47}$$

of the rotation vector if omitting $\mathrm{O}(\varepsilon^2)$, which describes the mean rotation of a cross-section of a rod or shell in an infinitesimal deformation.

Table 1. The angles χ_r, χ_{max}, α_w and α_{max} as functions of Θ (in degree)

Θ	0	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00
χ_w	0	9.766	19.53	29.28	39.03	48.75	58.46	68.13	77.76
α_w	—	8.161	8.254	8.413	8.644	8.958	9.367	9.893	10.56
$\chi_{\max}$	0	10.47	20.96	31.48	42.06	52.73	63.56	74.70	86.71
$\alpha_{\max}$	—	35.45	36.05	37.11	38.72	41.07	44.53	49.95	60.37

5. SMALL OR MODERATE STRAIN ACCOMPANIED BY LARGE ROTATION

In this section we demonstrate that $\mathbf{R}_w$ is a very good approximation of $\mathbf{Q}$ when the deformation is of small or moderate strain even though accompanied by large rotation. Thus, we propose to name $\mathbf{R}_w$ the large rotation tensor. Let $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ and $\mathbf{e} = \mathbf{U}-\mathbf{I}$ be the right Cauchy–Green deformation tensor and Biot strain tensor, respectively. The deformation is said to be of small strain if $|\mathbf{C}-\mathbf{I}| \ll 1$ or moderate strain if $|\mathbf{C}-\mathbf{I}|^2 \ll 1$. In a case of small or moderate strain, $\mathbf{e}$ can be approximately expressed in the form:

$$\mathbf{e} \approx \tfrac{1}{2}(\mathbf{C}-\mathbf{I}), \quad \text{or} \quad \mathbf{e} \cong \tfrac{1}{2}(\mathbf{C}-\mathbf{I}) - \tfrac{1}{8}(\mathbf{C}-\mathbf{I})^2. \tag{48}$$

Substituting $\mathbf{U} = \mathbf{I}+\mathbf{e}$ and the canonical form (36) of $\mathbf{Q}$ into $\mathbf{F} = \mathbf{QU}$ and (4), we can immediately obtain the following relations:

$$\mathbf{E} = (1-\cos\Theta)(\mathbf{p}\otimes\mathbf{p}-\mathbf{I}) + \tfrac{1}{2}(\mathbf{Qe}+\mathbf{eQ}^T), \tag{49a}$$

$$\operatorname{tr}\mathbf{E} = -2(1-\cos\Theta) + \operatorname{tr}\mathbf{Qe}, \tag{49b}$$

$$\mathbf{Ep} = \tfrac{1}{2}(\mathbf{Q}+\mathbf{I})\mathbf{ep}, \tag{49c}$$

$$\mathbf{p}\cdot\mathbf{Ep} = \mathbf{p}\cdot\mathbf{ep}, \tag{49d}$$

and

$$\mathbf{w} = \sin\Theta\mathbf{p} + \tfrac{1}{2}\sin\Theta(\mathbf{p}\operatorname{tr}\mathbf{e}-\mathbf{ep}) - \tfrac{1}{2}(1-\cos\Theta)\mathbf{p}\times\mathbf{ep} = T_\mathbf{p}\sin\Theta(\mathbf{p}-\zeta\mathbf{q}), \tag{50a}$$

$$\delta\mathbf{p} = \mathring{\mathbf{w}} - \mathbf{p} = (\cos\gamma - 1)\mathbf{p} - \sin\gamma\mathbf{q}, \tag{50b}$$

$$|\mathbf{w}| = T_\mathbf{p}\sec\gamma\sin\Theta, \tag{50c}$$

where

$$\mathbf{q} = \mathbf{p}\times\left(\cos\frac{\Theta}{2}\mathbf{p}\times\mathbf{ep} + \sin\frac{\Theta}{2}\mathbf{ep}\right)\Big/|\mathbf{p}\times\mathbf{ep}|, \tag{51a}$$

$$T_\mathbf{p} = 1 + \tfrac{1}{2}(\operatorname{tr}\mathbf{e} - \mathbf{p}\cdot\mathbf{ep}) = 1 + O(|\mathbf{e}|), \tag{51b}$$

$$\zeta = \left(2T_\mathbf{p}\cos\frac{\Theta}{2}\right)^{-1}|\mathbf{p}\times\mathbf{ep}| = O(|\mathbf{e}|), \tag{51c}$$

$$\cos\gamma = \mathring{\mathbf{w}}\cdot\mathbf{p} = 1 - \tfrac{1}{2}|\delta\mathbf{p}|^2, \tag{51d}$$

$$\tan\gamma = \zeta. \tag{51e}$$

From (50), we see that $\mathring{\mathbf{w}}$ would be quite close to $\mathbf{p}$ provided γ or ζ is small; and from (51b, c) we can have, under the restriction $|\mathbf{e}| =$ constant, the following inequalities:

$$|T_\mathbf{p} - 1| \leqslant \frac{\sqrt{2}}{2}|\mathbf{e}|, \tag{52a}$$

$$|\zeta| \leqslant \varepsilon_0 = \varepsilon_0(|\mathbf{e}|, \Theta) = \left[\frac{1}{\sqrt{8-4|\mathbf{e}|^2}}\sec\frac{\Theta}{2}\right]|\mathbf{e}|, \tag{52b}$$

or alternatively,

$$|\mathbf{e}| = \left[\sqrt{\frac{8}{1+4\varepsilon_0^2}}\cos\frac{\Theta}{2}\right]\varepsilon_0 \leqslant 2\sqrt{2}\varepsilon_0, \tag{52c}$$

$$|\Delta\mathbf{p}| = 2\left|\sin\frac{\gamma}{2}\right| \leqslant \tan\gamma = |\zeta| \leqslant \varepsilon_0. \tag{52d}$$

The parameter ε_0 plays a key role in the remaining analysis of the present section.

To compare $\chi_\mathbf{w}$ with Θ, noting that $\chi_\mathbf{r}$ is an analytical function of $\mathbf{r}$ in a neighborhood of $\mathbf{r} = \mathbf{p}$ because of $T_\mathbf{p} = 1 + O(|\mathbf{e}|) \neq 0$, we have

$$\chi_w = \Theta + d\chi_r|_{r=p} + \tfrac{1}{2} d^2\chi_r|_{r=p} + O(|\delta p|^3). \tag{53}$$

From (3) one can easily derive that

$$T_r d\chi_r = (\cos\chi_r \mathbf{w} + \sin\chi_r \mathbf{Er}) \cdot \delta\mathbf{r}, \tag{54a}$$

$$T_r d^2\chi_r = -2dT_r d\chi_r + (\sin\chi_r)\delta\mathbf{r} \cdot \mathbf{E}\delta\mathbf{r}. \tag{54b}$$

Furthermore, by using (39) and (49)–(54), we can write

$$\begin{aligned} T_p d\chi_r|_{r=p} &= (\cos\Theta\mathbf{w} + \sin\Theta\mathbf{Ep}) \cdot \delta\mathbf{p} = -\mathbf{w} \cdot \delta\mathbf{p} + [(1+\cos\Theta)\mathbf{w} \cdot \mathbf{p} + \sin\Theta]\mathbf{p} \cdot \delta\mathbf{p} \\ &= -|\mathbf{w}|\,|\delta\mathbf{p}|^2 + \{[(1+\cos\Theta)\cos\gamma - 1]\,|\,\mathbf{w}\,| + \sin\Theta\mathbf{p} \cdot \mathbf{ep}\}(\cos\gamma - 1) \\ &= -(1 + \tfrac{1}{2}\cos\Theta)\sin\Theta\,|\delta\mathbf{p}|^2 + O(|\,\delta\mathbf{p}\,|^3), \end{aligned} \tag{55a}$$

$$T_p d^2\chi_r|_{r=p} = \sin\Theta\delta\mathbf{p} \cdot \mathbf{E}\delta\mathbf{p} + O(|\delta\mathbf{p}|^3). \tag{55b}$$

Thus, from (49)–(55) we finally arrive at the relation

$$\chi_w = \Theta - \frac{1}{4}[(7-\cos\Theta)\sin\Theta]\,|\delta\mathbf{p}\,|^2 + O(|\,\delta\mathbf{p}\,|^3) = \Theta + O(\Theta\varepsilon_0^2). \tag{56}$$

As an interesting consequence of (56) and (37), it follows immediately

$$\chi_w \leqslant \Theta \leqslant \chi_{max}, \tag{57}$$

provided the deformation is of small or moderate strain accompanied by finite rotation. Reviewing (44) and Table 1 we see also that even though in a quite large strain, the relation (57) may still hold. However, up-to-now this is still a guess for a general finite deformation.

In contrast to the second order small deviation $O(\varepsilon_0^2)$ between χ_w and Θ, we can write only

$$\mathbf{R}_w = \mathbf{Q} + O(\Theta\varepsilon_0). \tag{58}$$

In order to give a better approximation of $\mathbf{Q}$ than (58), the following simple modification of $\mathbf{R}_w$ is proposed:

$$\tilde{\mathbf{Q}} = \cos\chi_w\mathbf{I} + \sin\chi_w\tilde{\mathbf{w}} \times \mathbf{I} + (1-\cos\chi_w)\tilde{\mathbf{w}} \otimes \tilde{\mathbf{w}} = \mathbf{Q} + O(\Theta\varepsilon_0^2), \tag{59a}$$

$$\tilde{\mathbf{w}} = |(\mathbf{F}+\mathbf{I})\mathbf{w}|^{-1}(\mathbf{F}+\mathbf{I})\mathbf{w} = \mathbf{p} + O(\Theta\varepsilon_0^2). \tag{59b}$$

The formulation of calculating $\mathbf{Q}$ given by (43), (44), (56), (58) and (59) constitutes a new basis for describing the kinematics of the non-linear theories of rods, plates, and shells having undergone small or moderate strain accompanied by finite rotation.

An abstract of the study given in this section above was presented by Xiong and Zheng (1989b). Some interesting applications of eqns (38), (39) and (56), can be seen in Xiong and Zheng (1993).

6. ON RATES OF MEAN ROTATIONS

Although tensorial expressions of the relative spin $\boldsymbol{\Omega}_p = \dot{\mathbf{Q}}\mathbf{Q}^T$ have been given by Dienes (1979), Guo (1984) and Zheng (1992), it is still of interest to discuss the rate of mean rotation for a while. Denote by $\mathbf{v} = \dot{\mathbf{x}}$, $\mathbf{G} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, $\mathbf{D} = (\mathbf{G}+\mathbf{G}^T)/2$ and $\boldsymbol{\Omega} = (\mathbf{G}-\mathbf{G}^T)/2$ the velocity vector, velocity gradient, rate of deformation tensor and material spin, respectively. The vorticity $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$: $\boldsymbol{\Omega} = \boldsymbol{\omega} \times \mathbf{I}$. From (3), we can immediately give the rate $\dot{\chi}_r$ of the generalized Cauchy's mean rotation angle χ_r with respect to a fixed rotation axis $\mathbf{r}$ in the form:

$$\dot{\chi}_r = \frac{(\mathrm{tr}\,\mathbf{\Gamma F})(\mathrm{tr}\,\mathbf{GF} - \mathbf{r}\cdot\mathbf{GFr}) - (\mathrm{tr}\,\mathbf{F} - \mathbf{r}\cdot\mathbf{Fr})(\mathrm{tr}\,\mathbf{\Gamma GF})}{(\mathrm{tr}\,\mathbf{\Gamma F})^2 + (\mathrm{tr}\,\mathbf{F} - \mathbf{r}\cdot\mathbf{Fr})^2}, \tag{60}$$

where $\mathbf{\Gamma} = \mathbf{r}\times\mathbf{I}$. In particular, if we take the current configuration κ of the deforming body as the reference configuration, i.e. $\mathbf{F} = \mathbf{I}$ at time t, then from (60) it follows

$$\dot{\chi}_r|_{\mathbf{F}=\mathbf{I}} = \boldsymbol{\omega}\cdot\mathbf{r}. \tag{61}$$

Thus, the vorticity vector $\boldsymbol{\omega}$ can exactly be explained as the spatial mean angular velocity.

Let $n(\varepsilon, \mathbf{x}) = \{\mathbf{x}+\delta\mathbf{x} : |\delta\mathbf{x}| < \varepsilon\}$, $0 < \varepsilon \ll 1$, be a ball neighborhood of $\mathbf{x}$ with respect to the current configuration κ. Consider a local rigid motion $(\mathbf{v}_x, \boldsymbol{\omega}_x)$ with $\mathbf{v}_x$ the linear velocity and ω_x the angular velocity at $\mathbf{x}$. Thus, the function

$$\int_{n(\varepsilon,\mathbf{x})} |(\mathbf{v}+\mathbf{G}\delta\mathbf{x}) - (\mathbf{v}_x + \boldsymbol{\omega}_x\times\delta\mathbf{x})|^2 = \frac{4\pi}{3}\varepsilon^2|\mathbf{v}-\mathbf{v}_x|^2 + \frac{4\pi}{15}\varepsilon^5(|\mathbf{D}|^2 + 2|\boldsymbol{\omega}-\boldsymbol{\omega}_x|^2) \tag{62}$$

describes the total square deviation between the actual local motion of $n(\varepsilon, \mathbf{x})$ and a rigid motion $(\mathbf{v}_x, \boldsymbol{\omega}_x)$ of $n(\varepsilon, \mathbf{x})$. Similar to Theorem 6, from (62) one can easily give the following theorem:

Theorem 7. (i) Among all rigid motions in the form $(\mathbf{v}_x = 0, \boldsymbol{\omega}_x = \omega\mathbf{r})$, the total square deviation (62) at $\omega = \dot{\chi}_r|_{\mathbf{F}=\mathbf{I}} = \boldsymbol{\omega}\cdot\mathbf{r}$ is the least possible; and (ii) among all rigid motions in the form $(\mathbf{v}_x, \boldsymbol{\omega}_x)$, the total square deviation (62) at $(\mathbf{v}_x, \boldsymbol{\omega}_x) = (\mathbf{v}, \boldsymbol{\omega})$ is the least possible.

7. SOME REMARKS ON ROTATION IN CONSTITUTIVE EQUATION AND ON CHOOSING A REFERENCE CONFIGURATION

To explain the effect of the finite rotation tensor $\mathbf{Q}$ in constitutive equations and the effect of choosing a reference configuration, we employ a simple constitutive model

$$\dot{\boldsymbol{\sigma}} = \psi(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{M}) \tag{63}$$

in absence of rotation, where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, $\mathbf{D}$ the rate of deformation tensor and $\mathbf{M}$ the structure tensor which characterizes the material symmetry (Zheng and Spencer, 1993; Zheng and Boehler, 1994). The use of $\mathbf{M}$ enables us to deal with ψ as an isotropic second-order symmetric tensor-valued function of $\boldsymbol{\sigma}$, $\mathbf{D}$, $\mathbf{M}$ for an anisotropic material whose material symmetry is characterized by $\mathbf{M}$.

Suppose that (63) is experimentally determined for any arotational deformation history (Zheng, 1990, 1992), that is, $\mathbf{Q}$ has been keeping to be the identity tensor $\mathbf{I}$. According to the constitutive equivalence principle established by Zheng (1990, 1992), the precise form of the constitutive equation corresponding to (63) for any general deformation history when the material is rotating is of the form:

$$\mathbf{Q}^{T\,D}\boldsymbol{\sigma}\mathbf{Q} = \psi(\mathbf{Q}^T\boldsymbol{\sigma}\mathbf{Q}, \mathbf{Q}^T\mathbf{D}\mathbf{Q}, \mathbf{M}), \tag{64}$$

where ${}^D\boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\boldsymbol{\Omega}_p - \boldsymbol{\Omega}_p\,\boldsymbol{\sigma}$ is the Dienes' rate of $\boldsymbol{\sigma}$, $\boldsymbol{\Omega}_p = \dot{\mathbf{Q}}\mathbf{Q}^T$ is the relative spin. Since ψ is isotropic, one can further arrange (64) into the following form:

$${}^D\boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\boldsymbol{\Omega}_p - \boldsymbol{\Omega}_p\boldsymbol{\sigma} = \psi(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{QMQ}^T). \tag{65}$$

This precise formulation shows that not only the relative spin $\boldsymbol{\Omega}_p$, but also the finite rotation tensor $\mathbf{Q}$ itself play important roles in modelling the mechanical behaviour of anisotropic material. We note that as an exceptional case, for an isotropic material, $\mathbf{M}$ disappears from (63)–(65) and the effect of the finite rotation tensor to the constitutive equation is indirect via $\boldsymbol{\Omega}_p$.

One may note that both strain and finite rotation tensor are relative concepts which depend upon the choice of a reference configuration. Let κ_0 and κ_0' be two reference configurations and κ the current configuration of the deformable body. The deformation and the deformation gradient with respect to κ_0 are denoted by $\mathbf{x}(\mathbf{X}, t)$ and $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$, to κ_0' by $\mathbf{x}(\mathbf{X}', t)$ and $\mathbf{F}' = \partial \mathbf{x}/\partial \mathbf{X}'$, respectively. Thus, we can write the following polar decompositions and relations:

$$\mathbf{F} = \mathbf{QU} = \mathbf{VQ}, \tag{66a}$$

$$\mathbf{F}' = \mathbf{Q}'\mathbf{U}' = \mathbf{V}'\mathbf{Q}', \tag{66b}$$

$$\mathbf{F}' = \mathbf{FP}, \mathbf{P} = \partial \mathbf{X}/\partial \mathbf{X}'. \tag{66c}$$

In general, $\mathbf{Q}$ is not the same as $\mathbf{Q}'$, and the solutions for any other measure of mean rotation with respect to κ_0 and κ_0' are different.

The constitutive equation of a solid is, in most cases, proposed and experimentally determined with respect to an undeformed (or undistorted) configuration, say κ_0, as the reference configuration. From the geometric compatible condition we know that another reference configuration κ_0' is also undeformed if and only if there is a constant rotation tensor $\mathbf{R}_0$ so that $\mathbf{P} = \mathbf{R}_0$ and thus

$$\mathbf{Q}' = \mathbf{QR}_0, \quad \mathbf{V}' = \mathbf{V}. \tag{67}$$

In other words, to effectively utilize the constitutive equation established with respect to an undeformed configuration κ_0, we should choose this configuration itself or other configuration κ_0' which differs in a rigid translation and a rigid rotation from κ_0 as the reference configuration. Since from (67) we can obtain

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma}, \quad \mathbf{D}' = \mathbf{D}, \quad \boldsymbol{\Omega}_p' = \boldsymbol{\Omega}_p, \quad \mathbf{M}' = \mathbf{R}_0^{\mathrm{T}}\mathbf{M}\mathbf{R}_0, \tag{68}$$

the constitutive functional Ψ is thus unaltered in the sense

$$^{D}\boldsymbol{\sigma}' = \boldsymbol{\sigma}' + \boldsymbol{\sigma}'\boldsymbol{\Omega}_p' - \boldsymbol{\Omega}_p'\boldsymbol{\sigma}' = \Psi(\boldsymbol{\sigma}', \mathbf{D}', \mathbf{Q}'\mathbf{M}'\mathbf{Q}'^{T}). \tag{69}$$

The reader may work on a more general constitutive equation represented in the frame of internal variables and structure tensors (Zheng and Boehler, 1994) and conclude again the importance of both $\boldsymbol{\Omega}_p$ and $\mathbf{Q}$ based upon the constitutive equivalence principle (Zheng, 1990, 1992).

8. ON THE GLOBAL MEAN ROTATIONS AND GLOBAL KINETICS

A proper global description of the mean rotation of a deforming body B is of practical interest in some important cases, for example, the problem of stability of satellites. Let κ_0 and κ denote the primarily undeformed configuration and the current (at time t) configuration of the deforming body B, respectively, M be the mass of B, and take mass element $\mathrm{d}m$ as the measures of both κ_0 and κ. Then, the moment of inertia tensors of κ_0 and κ with respect to their mass centers $\mathbf{X}_c$ and $\mathbf{x}_c$:

$$\mathbf{X}_c = M^{-1}\int_{\kappa_0} \mathbf{X}\,\mathrm{d}m, \quad \mathbf{x}_c = \mathrm{M}^{-1}\int_{\kappa} \mathbf{x}\,\mathrm{d}m \tag{70}$$

are

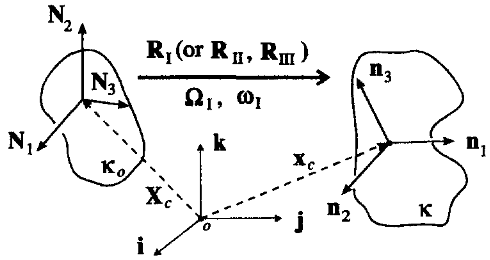


Fig. 6. The inertia principal axes, inertia rotations, inertia spin, and inertia vorticity.

$$\mathbf{I}_c = \int_{\kappa_0} \{ |\mathbf{X}-\mathbf{X}_c|^2\mathbf{I} - (\mathbf{X}-\mathbf{X}_c)\otimes(\mathbf{X}-\mathbf{X}_c)\}\,\mathrm{d}m = \sum_{k=1}^{3} I_k\mathbf{N}_k\otimes\mathbf{N}_k, \tag{71a}$$

$$\mathbf{i}_c = \int_{\kappa} \{ |\mathbf{x}-\mathbf{x}_c|^2\mathbf{I} - (\mathbf{x}-\mathbf{x}_c)\otimes(\mathbf{x}-\mathbf{x}_c)\}\,\mathrm{d}m = \sum_{k=1}^{3} i_k\mathbf{n}_k\otimes\mathbf{n}_k. \tag{71b}$$

Without loss of generality, we suppose that the inertia principal axes $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ of $\mathbf{I}_c$ and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ of $\mathbf{i}_c$ are both right-handed. Thus, from the transformations between $\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ we can induce the following three rotation tensors (Fig. 6):

$$\mathbf{R}_I = \sum_{k=1}^{3} \mathbf{n}_k\otimes\mathbf{N}_k, \mathbf{R}_{II} = \mathbf{R}_I\mathbf{T}, \mathbf{R}_{III} = \mathbf{R}_I\mathbf{T}^2, \tag{72}$$

where $\mathbf{T}$ indicates the rotation tensor about the direction $\mathbf{N}_0 = \mathbf{N}_1+\mathbf{N}_2+\mathbf{N}_3$ through the angle $2\pi/3$, i.e.

$$\mathbf{T} = \tfrac{1}{2}(-\mathbf{I}+\sqrt{3}\mathbf{N}_0\times\mathbf{I}+3\mathbf{N}_0\otimes\mathbf{N}_0). \tag{73}$$

We call $\mathbf{R}_I$, $\mathbf{R}_{II}$, and $\mathbf{R}_{III}$ the global inertia rotation tensors of the deforming body B. However, all these three inertia rotation tensors possess the same so-called global inertia spin $\boldsymbol{\Omega}_I$ and global inertia vorticity $\boldsymbol{\omega}_I$:

$$\boldsymbol{\Omega}_I = \dot{\mathbf{R}}_I\mathbf{R}_I^{\mathrm{T}} = \dot{\mathbf{R}}_{II}\mathbf{R}_{II}^{\mathrm{T}} = \dot{\mathbf{R}}_{III}\mathbf{R}_{III}^{\mathrm{T}} = \boldsymbol{\omega}_I\times\mathbf{I}. \tag{74}$$

From (74) we have

$$\dot{\mathbf{n}}_k = \dot{\mathbf{R}}_I\mathbf{N}_k = \boldsymbol{\Omega}_I\mathbf{n}_k = \boldsymbol{\omega}_I\times\mathbf{n}_k. \tag{75}$$

The total linear momentum $\mathbf{l}$ and angular momentum $\mathbf{j}$ of the deforming body B are respectively

$$\mathbf{l} = \int_{\kappa} \mathbf{v}\,\mathrm{d}m, \tag{76a}$$

$$\mathbf{j} = \int_{\kappa} \mathbf{x}\times\mathbf{v}\,\mathrm{d}m = \mathbf{x}_c\times\mathbf{l}+\mathbf{j}_c, \quad \left[\mathbf{j}_c = \int_{\kappa} (\mathbf{x}-\mathbf{x}_c)\times\mathbf{v}\,\mathrm{d}m\right]. \tag{76b}$$

Let $\mathbf{f}$ and $\mathbf{m}$ be the total force and moment of force applied on the current configuration κ. The laws of conservation of linear and angular momentums can be expressed in the forms:

$$\mathbf{f} = \dot{\mathbf{I}} = M\dot{\mathbf{v}}_c = M\ddot{\mathbf{x}}_c, \tag{77}$$

$$\mathbf{m} - \mathbf{x}_c \times \mathbf{f} = \dot{\mathbf{j}} - \mathbf{x}_c \times \mathbf{f} = \dot{\mathbf{j}}_c = \dot{\mathbf{i}}_c \omega_c + \mathbf{i}_c \dot{\omega}_c, \tag{78}$$

where the global velocity $\mathbf{v}_c$ and global vorticity ω_c are defined by

$$\mathbf{I} = M\mathbf{v}_c = M\dot{\mathbf{x}}_c, \tag{79a}$$

$$\mathbf{j}_c = \mathbf{i}_c \omega_c. \tag{79b}$$

In general, the traditional global vorticity ω_c is not equal to the global inertial one ω_I. Since from (75) and (78) we have

$$\dot{\mathbf{i}}_c = \frac{\mathrm{d}_I}{\mathrm{d}t}\mathbf{i}_c + \omega_I \times \mathbf{i}_c - \mathbf{i}_c \times \omega_I, \tag{80a}$$

$$\frac{\mathrm{d}_I}{\mathrm{d}t}\mathbf{i}_c = \dot{\mathbf{i}}_k \mathbf{n}_k \otimes \mathbf{n}_k, \tag{80b}$$

substituting (80a) into (78) yields

$$\mathbf{i}_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_c\right) + \omega_I \times \mathbf{i}_c \omega_c + \left\{\left(\frac{\mathrm{d}_I}{\mathrm{d}t}\mathbf{i}_c\right)\omega_c - \mathbf{i}_c(\omega_I \times \omega_c)\right\} = \mathbf{m} - \mathbf{x}_c \times \mathbf{f}. \tag{81}$$

Hence, the global inertia spin $\mathbf{\Omega}_I$ or global inertia vorticity ω_I plays an key role in the global kinetic equation (81). Besides, from (81) we see that if and only if

$$\omega_I = \omega_c, \tag{82a}$$

$$\frac{\mathrm{d}_I}{\mathrm{d}t}\mathbf{i}_c = \mathbf{0}, \tag{82b}$$

(81) degenerates into the traditional rigid kinetic equation with respect to the conservation of angular momentum.

Finally, we point out that more global measures of the mean rotation can also be introduced in terms of the local measures of the mean rotation. For example, consider

$$\bar{\chi}_{\mathbf{r}}(B) = M^{-1}\int_{\mathbf{X}\in B} \chi_{\mathbf{r}}(\mathbf{X})\,\mathrm{d}m, \tag{83a}$$

$$\bar{\mathbf{F}}(B) = M^{-1}\int_{\mathbf{X}\in B} \mathbf{F}(\mathbf{X})\,\mathrm{d}m = \bar{\mathbf{Q}}(B)\bar{\mathbf{U}}(B), \quad [\bar{\mathbf{U}}(B)^2 = \bar{\mathbf{F}}(B)^T\bar{\mathbf{F}}(B)]. \tag{83b}$$

We name $\bar{\chi}_{\mathbf{r}}(B)$ the global Cauchy's mean rotation angle about the axis $\mathbf{r}$, and $\bar{\mathbf{Q}}(B)$ the global finite rotation tensor. Let $\bar{\mathbf{R}}_{\mathbf{r}}(B)$ denote the "Cauchy's mean rotation tensor" about $\mathbf{r}$ as if $\bar{\mathbf{F}}(B)$ were the "deformation gradient". One can easily prove that at $\bar{\mathbf{Q}}(B)$ or $\bar{\mathbf{R}}_{\mathbf{r}}(B)$ the total mean deviation

$$\int_{\mathbf{X}\in B} \Delta(\mathbf{R}, U; \mathbf{X})\,\mathrm{d}m = 3M + \mathrm{tr}\int_{\mathbf{X}\in \mathbf{B}} \mathbf{F}^T\mathbf{F}\,\mathrm{d}m - 2M\,\mathrm{tr}\,\mathbf{R}^T\bar{\mathbf{F}}(B) \tag{84}$$

as a function of $\mathbf{R}$ in the domain $\mathfrak{R}$ or $\mathfrak{R}_r$, respectively, is the least possible. In (84) the U is the full direction set and the $\Delta(\mathbf{R}, U)$ defined in (20) has been rewritten into $\Delta(\mathbf{R}, U; \mathbf{X})$ in order to emphasize the dependence on $\mathbf{X}$. However, the rotation angle of $\bar{\mathbf{R}}_{\mathbf{r}}(B)$ and the $\bar{\chi}_{\mathbf{r}}(B)$ are usually different if the deformation is not homogeneous.

9. CONCLUSIONS

The original Cauchy's measure of mean rotation of a deformable body possesses a serious defect and is not always equal to the (generalized) Cauchy's mean rotation angle formulated by Zheng and Hwang (1988, 1992). The concept of mean rotation tensor proposed in this paper is a suitable modification and generalization of the classical concept of Cauchy's mean rotation, because in the former the essential defects appeared in the latter have been removed and more measures of mean rotation can have analytical forms. In this frame, the most significant rotation measure, i.e. the finite rotation tensor Q in the polar decomposition of the deformation gradient has been associated with some new significant geometric explanations.

As the deformation is of small or moderate strain accompanied by finite rotation, the so-called large rotation tensor $\mathbf{R}_w$, which can be simply calculated, is a good approximation of the finite rotation tensor $\mathbf{Q}$. This may be applied to formulate new kinematics for non-linear theories of rods, plates, and shells. On the other hand, the mean rotation tensor of a cross-section of a rod, plate, or shell has also been investigated. In general, $\mathbf{Q}$ is neither $\mathbf{R}_w$, nor the maximum Cauchy's mean rotation tensor $\mathbf{R}_{\max}$.

A short discussion about rates of mean rotation is provided and two new geometric meanings of the vorticity vector is revealed. We also consider the importance of both the relative spin $\mathbf{\Omega}_p = \dot{\mathbf{Q}}\mathbf{Q}^T$ and the finite rotation tensor $\mathbf{Q}$ in constitutive equations of anisotropic materials, and check the effect of choosing a reference configuration. This paper concludes with a short investigation on the global measures of mean rotation of a deforming body. Finally, we present the global kinetic equations in terms of global inertia spin or global inertia vorticity which are associated with the rate of the inertia rotation tensor, a global measure of mean rotation in a sense. We also propose two other global measures of mean rotation.

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REFERENCES

Bowen, R. M. and Wang, C.-C. (1976). *Introduction to Vectors and Tensors*. Plenum, New York.
Cauchy, A.-L. (1841). Mémoire sur les dilatations, les condensations et les rotations produits par un changement de forme dans un système de points matérials. *Ex. d'an. Phys. Math.* **2,** 302–330 = Oeuvres (2) **12,** 343–377.
Dafalias, Y. F. (1983). Corotational rates for kinematic hardening at large plastic deformations. (ASME) *J. Appl. Mech.* **50,** 561–565.
Dafalias, Y. F. (1987). Issues on the constitutive formulation at large elastoplastic deformations, Part 1 : kinematics. *Acta Mech.* **69,** 119–138.
Dafalias, Y. F. (1988). Issues on the constitutive formulation at large elastoplastic deformations, Part 2 : kinetics. *Acta Mech.* **73,** 121–146.
Dienes, J. K. (1979). On the analysis of rotation and stress rate in deforming bodies. *Acta Mech.* **32,** 217–232.
Dienes, J. K. (1987). A discussion of material rotation and stress rate. *Acta Mech.* **65,** 1–11.
Grioli, G. (1940). Una proprieta de minimo nella cinematica delle deformazioni finite. *Boll. Un. Mat. Ital.* **2,** 452–455.
Guo, Z.-H. (1981). Representations of orthogonal tensors. *SM Archive* **4,** 451–466.
Guo, Z.-H. (1984). Rates of stretch tensors. *J. Elasticity* **14,** 263–27.
Gurtin, M. E. (1981). *An Introduction to Continuum Mechanics*. Academic Press, New York.
Hoger, A. and Carlson, D. E. (1984). Determination of the stretch and rotation in the polar decomposition of the deformation gradient. *Quart. Appl. Math.* **42,** 113–117.
Martins, L. C. and Podio-Guidugli, P. (1980). An elementary proof of the polar decomposition theorem. *Amer. Math. Month.* **87,** 288–290.
Martins, L. C. and Podio-Guidugli, P. (1989). On the local measures of mean rotation in continuum mechanics. AMS (MOS) Subject Classification : 73B15.
Martins, L. C. and Podio-Guidugli, P. (1992). On the local measures of mean rotation in continuum mechanics. *J. Elasticity* **27,** 267–279.
Marzano, S. (1987). On mean rotations in finite deformation. *Meccanica* **22,** 223–226.
Noll, W. (1958). A mathematical theory of the mechanical behavior of continuous media. *Arch. Rat. Mech. Anal.* **2,** 197–226.
Novozhilov, V. V. (1948). *Foundation of the Nonlinear Theory of Elasticity* (in Russian) ; (English translation, 1953). Graylock Press, Rochester, New York.
Truesdell, C. and Toupin, R. A. (1960). The classical field theories. In *Encyclopedia of Physics* (Edited by S. Flügge). Vol. III/1, pp. 226–902. Springer-Verlag, Berlin.

Xiong, Z.-H. and Zheng, Q.-S. (1988). General algorithms for the polar decomposition and strains. *Acta Mech. Sinica* **4,** 175–181.

Xiong, Z.-H. and Zheng, Q.-S. (1989a). Canonical representations and degree of freedom formulae of orthogonal tensors in *n*-dimensional euclidean space. *Appl. Math. Mech.* **10,** 93–101; (in Chinese) **10,** 85–93.

Xiong, Z.-H. and Zheng, Q.-S. (1989b). Moderate strain accompanied by large rotation (in Chinese). In *Proc. MMM-III* (*3rd Modern Math. Mech.*), pp. 498–503. Science Press, Beijing.

Xiong, H.-E. and Zheng, Q.-S. (1993). Approach to approximate analysis of deformation and strain. *Acta Mech. Solida Sinica* **6,** 261–273.

Zheng, Q.-S. (1987). On the mean rotation of a finite deformable body. In *Proc. MMM-II* (*2nd Modern Math. Mech*), pp. 76–79. Shanghai.

Zheng, Q.-S. (1989). Some fundamental problems in continuum mechanics. Ph.D. Thesis, Tsinghua University.

Zheng, Q.-S. (1990). The formulation of the generalized constitutive equations from their irotational forms for simple materials. *Chinese Science Bulletin* (in Chinese) **36,** 1751–1754; (English translation, 1991) **37,** 1217–1221.

Zheng, Q.-S. (1992). On the generalization of constitutive laws from their arotational forms. *Acta Mech.* **91,** 97–105.

Zheng, Q.-S. (1993). On Cauchy's mean rotation—closure. (Trans. ASME) *J. Appl. Mech.* **60,** 252–253.

Zheng, Q.-S. and Boehler, J.P. (1994). The description, classification, and reality of material and physical symmetries. *Acta Mech.* **102,** 73–89.

Zheng, Q.-S. and Hwang, K.-C. (1988). Cauchy's mean rotation. *Chinese Sci. Bull.* (in Chinese) **33,** 1705–1707; (English translation, 1989) **34,** 897–901.

Zheng, Q.-S. and Hwang, K.-C. (1992). On Cauchy's mean rotation. (Trans. ASME) *J. Appl. Mech.* **59,** 505–510.

Zheng, Q.-S. and Spencer, A.J.M. (1993). Tensors which characterize anisotropies. *Int. J. Engng Sci.* **31,** 679–793.